Lecture 07
13.1: Curves in space and their derivatives

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February 1, 2019

Things to note

Exam 1 is on Monday, February 11 (10 days away).

Quiz 04 will be on Wednesday, February 6.

Friday, February 8 will be a review day with no quiz.

Last few classes

Distance from S to a line with point P and direction vector $\vec{\mathbf{v}}$:

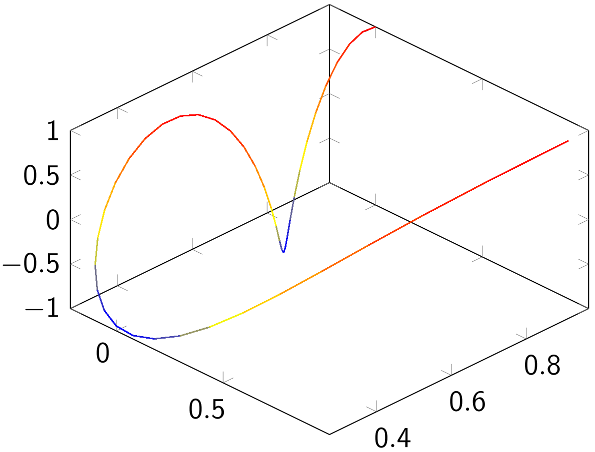
$$\frac{\|\overrightarrow{PS} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}$$

Distance from S to a plane with point P and normal vector $\vec{\mathbf{n}}$:

$$\frac{|\overrightarrow{PS} \cdot \vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|}$$

Only first formula will be on today's quiz.

Curves in space



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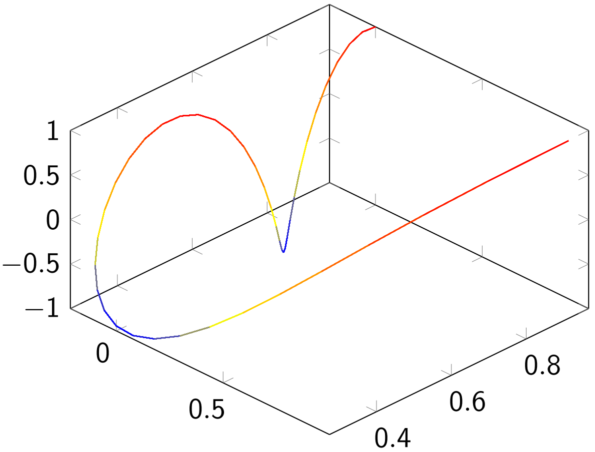

Can think of such a curve as the path of a particle moving in three dimensions.

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Definition

A <u>*vector-valued function*</u> *is a function*

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where f, g, *and* h *are real-valued functions.*

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We have already seen an example of a space curve: lines.

Example

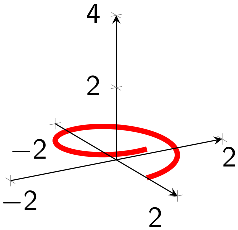
The function $\vec{\mathbf{r}}(t) = \langle 1 + t, -2t, 3 + 5t \rangle$ is a space curve (in this case, also a line).

Example

There are also more complex examples, such as the helix below.

Example

$\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$, *graphed from* $t = 0$ *to* $t = 2\pi$.



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and the limit exists if all three limits exist.

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Let $\vec{\mathbf{r}}(t) = \left\langle \frac{\sin(t)}{t}, e^{-t} + 5, \frac{t^2}{2t^2 - 4} \right\rangle$. *What is* $\lim_{t\to 0} \vec{\mathbf{r}}(t)$*?*

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Using our knowledge of limits from Calculus 1, we get $\lim_{t\to 0} \vec{\mathbf{r}}(t) = \langle 1, 6, 0\rangle$.

Continuity

Continuity is also defined in each component.

Definition

Let $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$*. Then* $\vec{\mathbf{r}}(t)$ *is continuous at* $t = t_0$ *if*

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that is, it is both the case that the limit exists and is equal to $\vec{\mathbf{r}}(t_0)$*.*

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Consider the helix, $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$*.Each function* $\cos(t), \sin(t)$*, and* t *is continuous on* $\mathbb{R}$*. Thus* $\vec{\mathbf{r}}(t)$ *is continuous.*

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The example $\vec{\mathbf{r}}(t) = \left\langle \frac{\sin(t)}{t}, e^{-t} + 5, \frac{t^2}{2t^2 - 4} \right\rangle$ would not be continuous at $t = 0$, however, since $\vec{\mathbf{r}}(0)$ is not defined.

Derivatives

We can also take derivatives of vector-valued functions. Once again, this simply happens in each component.

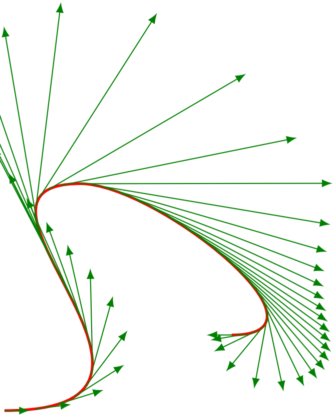
Definition

Let $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle$. $\vec{\mathbf{r}}$ *is differentiable at* $t = t_0$ *if* f, g *and* h *are differentiable at* t_0. *In this case,*

$$\vec{r}'(t) = \frac{d\vec{\mathbf{r}}}{dt} = \left\langle \frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt} \right\rangle.$$

Derivative vectors

We can visualize $\frac{d\vec{\mathbf{r}}}{dt}$ geometrically as the tangent vector to the space curve.



Differentiation Rules (pg. 756)

Let $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ be vector-valued functions. Let $\vec{\mathbf{C}}$ be a constant vector. Let $k \in \mathbb{R}$ be a scalar, and let $f(t)$ be a differentiable real-valued function. Then the following rules hold.

1. $\frac{d}{dt}\left[\vec{\mathbf{C}}\right] = \vec{\mathbf{0}}$

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$\frac{d}{dt}\left[f(t)\vec{\mathbf{u}}(t)\right] = \frac{df}{dt}\vec{\mathbf{u}}(t) + f(t)\frac{d\vec{\mathbf{u}}}{dt}$

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5. $\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \cdot \vec{\mathbf{v}}(t)\right] = \frac{d\vec{\mathbf{u}}}{dt} \cdot \vec{\mathbf{v}}(t) + \vec{\mathbf{u}}(t) \cdot \frac{d\vec{\mathbf{v}}}{dt}$
6. $\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{v}}(t)\right] = \frac{d\vec{\mathbf{u}}}{dt} \times \vec{\mathbf{v}}(t) + \vec{\mathbf{u}}(t) \times \frac{d\vec{\mathbf{v}}}{dt}$
7. $\frac{d}{dt}\left[\vec{\mathbf{u}}(f(t))\right] = f'(t)\vec{\mathbf{u}}'(f(t))$

Notation

If we think of $\vec{\mathbf{r}}(t)$ as the path of a particle in space, we use the notation

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}}{dt} \text{ and } \vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2}.$$

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The quantity $\|\vec{\mathbf{v}}(t)\|$ represents the speed of the particle at time t. Notice that this is a real-valued function.

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Find $\vec{\mathbf{v}}(t)$, $\|\vec{\mathbf{v}}(t)\|$, *and* $\vec{\mathbf{a}}(t)$ *for the helix* $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$.

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We have $\vec{\mathbf{v}}(t) = \langle -\sin(t), \cos(t), 1 \rangle$ and $\vec{\mathbf{a}}(t) = \langle -\cos(t), -\sin(t), 0 \rangle$. We calculate

$$\|\vec{\mathbf{v}}\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}.$$